Reynolds' Parametricity

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Based on joint work with Neil Ghani, Fredrik Nordvall Forsberg, Federico Orsanigo, and Tim Revell

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Course Outline

Topic: Reynolds' theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds' theory
- generalize Reynolds' construction to very general models

- **Lecture 1: Reynolds' theory of parametricity for System F**
- **Lecture 2: Introduction to fibrations**
- **Lecture 3: A bifibrational view of parametricity**
- **Lecture 4: Bifibrational parametric models for System F**

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- Throughout, let $\mathsf{Rel}(U)$ be an equality preserving arrow fibration and ∀-fibration

Fibrational Semantics of Types

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- **No definition for $[\![\Delta \vdash \tau]\!]$ on morphisms is needed because the domain of $[\![\Delta \vdash \tau]\!]$ is discrete**

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$$\forall : (|\mathsf{Rel}(U)|^{n+1} \rightarrow_{\mathsf{Eq}} \mathsf{Rel}(U)) \rightarrow (|\mathsf{Rel}(U)|^{n} \rightarrow_{\mathsf{Eq}} \mathsf{Rel}(U))$$

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- Indeed, the very *existence* of $\forall$ in a $\forall$-fibration requires that if F is equality preserving then so is $\forall F$

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- If U is faithful, then the $\forall$-fibration requirement can be reformulated in terms of more basic concepts using opfibrational structure of U

Fibrational Semantics of Terms - The Set Up

- In a CCC, for all X and Y, there is an object $X \Rightarrow Y$ and a isomorphism

$$\lambda : \mathsf{Hom}(W \times X, Y) \cong \mathsf{Hom}(W, X \Rightarrow Y)$$

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- **In a $\forall$-fibration, for every F and G, there is are isomorphisms**

$$\varphi_n : \mathsf{Hom}(F \circ \pi_n, G) \cong \mathsf{Hom}(F, \forall_n G)$$

that are natural in n

Fibrational Semantics of Terms - term variables

Define fibred natural transformations

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- **This specializes to our Set interpretation of variables**

Fibrational Semantics of Terms - term abstractions

- If

$$\frac{\Delta;\ \Gamma, x : \tau_1 \vdash t : \tau_2}{\Delta;\ \Gamma \vdash \lambda x.t : \tau_1 \to \tau_2}$$

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- This is sensible because

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- This specializes to our **Set** interpretation of term applications

Fibrational Semantics of Terms - type abstractions

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$$\frac{\Delta, \alpha; \Gamma \vdash t : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.t : \forall\alpha.\tau}$$

then

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- This is sensible because α is not free in Γ, so

$$\begin{aligned} [\![\Delta, \alpha; \Gamma \vdash t : \tau]\!] &: [\![\Delta, \alpha \vdash \Gamma]\!] \to [\![\Delta, \alpha \vdash \tau]\!] \\ &= [\![\Delta \vdash \Gamma]\!] \circ \pi_{|\Delta|} \to [\![\Delta, \alpha \vdash \tau]\!] \end{aligned}$$

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Validating β- and η-Rules

- **Our model is sensible by construction**

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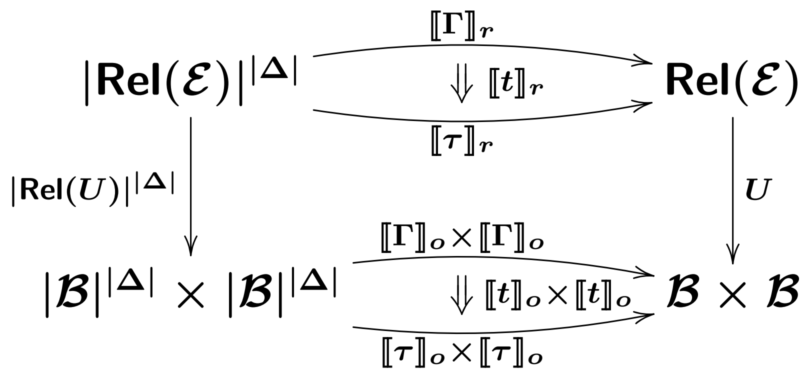

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In particular, for every fibration $U : \mathcal{E} \to B$ whose relations fibration is an equality preserving arrow fibration and a forall fibration, for every System F type $\Delta \vdash \tau$ and term $\Delta; \Gamma \vdash t : \tau$, we get:

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- **This hits the sweet spot between the simplicity and "light structure" of functorial models and the ability to prove expected key results**

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- **The PER model of Bainbridge et al. is also an instance (if bifibrations are understood as internal to the category of ω-sets)**

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- Ex: Using non-standard relations, we can construct a model of "multi-valued parametricity" over a constructively completely distributive complete non-trivial lattice of truth values

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- At WadlerFest, Neil Ghani, Fredrik Nordvall Forsberg, and Federico Orsanigo developed a proof-relevant version of our framework
- Clément Aubert, Fredrik Nordvall Forsberg, and I are working on extending our framework to a polymorphic calculus with computational effects (System F with effect-free constants and algebraic operations in the style of Plotkin and Power's effectful simply-typed calculus λ_c)

References

- **Functorial Polymorphism. E.S. Bainbridge, P.J. Freyd, A. Scedrov, and P. Scott. Theoretical Computer Science, 1990. [Gives a functorial semantics of polymorphism]**
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- **Categorical models for Abadi and Plotkin's logic for parametricity. L. Birkedal and R. Møgelberg. Mathematical Structures in Computer Science, 2005. [Constructs sophisticated models of parametricity and its logical structure. Also argues that not all expected consequences hold in Ma and Reynolds' framework]**
- **Parametric limits. B. Dunphy and U. Reddy. LICS'04. [First model to mix fibrations with reflexive graphs, but obtains existence of initial algebras only for strictly positive functors]**
- **And *many, many* more...**