Lecture 2 Exercises

1. Prove that $cod : \mathsf{Set}^{\to} \to \mathsf{Set}$ is a fibration.

2. Generalize the definitions of the slice and arrow categories Set/I and $\mathsf{Set}^{\to}$ to define the slice category $\mathcal{B}/I$ and the arrow category $\mathcal{B}^{\to}$ for an arbitrary category $\mathcal{B}$ and an object I of $\mathcal{B}$. Then generalize the definition of the codomain functor $cod : \mathsf{Set}^{\to} \to \mathsf{Set}$ to define the codomain functor $cod : \mathcal{B}^{\to} \to \mathcal{B}$.

3. Let $cod : \mathcal{B}^{\to} \to \mathcal{B}$.

   a) Show that the fibre $\mathcal{B}_I$ over an object I in $\mathcal{B}$ with respect to cod is the slice category $\mathcal{B}/I$.

   b) Show that the cartesian morphisms with respect to cod in $\mathcal{B}^{\to}$ coincide with pullback squares in $\mathcal{B}$.

   c) Conclude that cod is a fibration iff $\mathcal{B}$ has pullbacks. In this case, cod is called the *codomain fibration on* $\mathcal{B}$.

4. Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration.

   a) Show that every morphism in $\mathcal{E}$ factors as a vertical morphism followed by a cartesian morphism.

   b) Show that a cartesian morphism over an isomorphism is an isomorphism. Conclude that, in particular every vertical cartesian morphism is an isomorphism.

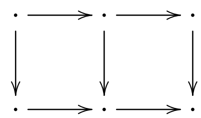
5. Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration.

   a) Show that all isomorphisms in $\mathcal{E}$ are cartesian. Conclude that, in particular, all identity morphisms in $\mathcal{E}$ are cartesian.

   b) Show that if $f : X \to Y$ and $g : Y \to Z$ are cartesian, then $g \circ f : X \to Z$ is also cartesian.

   c) Show that if $g : Y \to Z$ and $g \circ f : X \to Z$ are cartesian, then $f : X \to Y$ is also cartesian.

6. Consider the following diagram



Use parts 2 and 3 of Problem 5 to prove the following Pullback Lemmas:

a) If the left and right squares are pullback squares, then so is the outer square.

b) If the outer square and the right square are pullback squares, then so is the left square.

7. a) Prove that if U is a fibration, then so is $|U|$.

b) Prove that if U is a fibration, then so is U^n for any natural number n.