

# Semantics of Advanced Data Types

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June 14, 2021

# Why Study Semantics of Advanced Data Types?

- A language's type system allows us to express correctness properties of its programs. ("Well-typed programs don't go wrong.")
- Data types are an important part of any type system.
- Fancier data types let us express more sophisticated correctness properties.
- Type-checking provides guarantees of correctness with respect to these properties.
- In this course:



- We ask (and answer!)
  - What correctness properties can each class of data types express?
  - What models can we build to understand each class of data types?
  - What do properties of these models say about how we can compute with, and reason about, programs involving each class of data types?

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# Course Outline

**Lecture 1:** Syntax and semantics of ADTs and nested types

Lecture 2: Syntax and semantics of GADTs

Lecture 3: Parametricity for ADTs and nested types

Lecture 4: Parametricity for GADTs

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Lecture 1: Syntax and semantics of ADTs and nested types

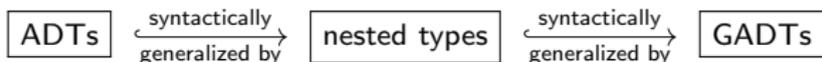
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Lecture 4: Parametricity for GADTs

# Lecture 1:

## Syntax and Semantics of ADTs and Nested Types



Assumption: Basic familiarity with categories, functors, natural transformations.

# Syntax of ADTs (I)

- Booleans

```
data Bool : Set where
  false : Bool
  true  : Bool
```

- Natural numbers

```
data Nat : Set where
  zero : Nat
  suc  : Nat → Nat
```

- Lists

```
data List (A : Set) : Set where
  [] : List A
  _::_ : A → List A → List A
```

- Binary trees

```
data Tree (A : Set) (B : Set) : Set where
  leaf : A → Tree A B
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## Syntax of ADTs (II)

- The only instance of the data type being defined that appears in the type of a constructor for that data type is the same one being defined
- So an ADT defines a family of inductive types, one for each choice of parameters.
- The general form of an ADT is

$$\begin{aligned} &\text{data } D A_1 \dots A_n : \text{Set where} \\ & \quad c_1 : T_{11} \rightarrow \dots \rightarrow T_{1j_1} \rightarrow D A_1 \dots A_n \\ & \quad \dots \\ & \quad c_k : T_{k1} \rightarrow \dots \rightarrow T_{kj_k} \rightarrow D A_1 \dots A_n \end{aligned}$$

- Agda also imposes a strict positivity requirement on the types of  $c_1, \dots, c_k$ : Either
  - $T_{ij}$  is not inductive and does not mention  $D$or
  - $T_{ij}$  is inductive and has the form

$$C_1 \rightarrow \dots \rightarrow C_p \rightarrow D A_1 \dots A_n$$

where  $D$  does not occur in any  $C_i$ .

- Strict positivity
  - $\implies$  no negative occurrences of  $D$  in the argument types of its constructors
  - $\implies$   $D$  can be interpreted as the least fixpoint of a functor.

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# Category Theory Interlude (I)

- A category  $\mathcal{C}$  comprises
  - a class  $ob(\mathcal{C})$  of *objects*
  - for each  $X, Y \in ob(\mathcal{C})$ , a class  $Hom_{\mathcal{C}}(X, Y)$  of *morphisms* from  $X$  to  $Y$
  - for each  $X \in ob(\mathcal{C})$ , an *identity* morphism  $id_X \in Hom_{\mathcal{C}}(X, X)$
  - a *composition* operator  $\circ$  assigning to each pair of morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , the composite morphism  $g \circ f : X \rightarrow Z$
- The identity morphisms are expected to behave like identities: if  $f : X \rightarrow Y$  then  $f \circ id_X = f = id_Y \circ f$ .
- Composition is associative.
- We write  $X : \mathcal{C}$  rather than  $X \in ob(\mathcal{C})$  and  $f : X \rightarrow Y$  rather than  $f \in Hom_{\mathcal{C}}(X, Y)$ .
- We will restrict attention to the category *Set* for now.

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## Category Theory Interlude (II)

- If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  comprises
  - a function  $F$  from  $ob(\mathcal{C})$  to  $ob(\mathcal{D})$ , together with
  - a function  $map_F$  from  $Hom_{\mathcal{C}}(X, Y)$  to  $Hom_{\mathcal{D}}(FX, FY)$
- A functor must preserve the fundamental structure of a category. This means that  $map_F$  must preserve identities and composition:

$$\begin{aligned} map_F g \circ map_F f &= map_F (g \circ f) \\ map_F id_X &= id_{FX} \end{aligned}$$

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# Functorial Semantics for ADTs: Overview

- Each ADT has an underlying functor  $F$  because of strict positivity.
- Kelly's Transfinite Construction of Free Algebras (TFCA) constructs free (i.e., initial) algebras for these functors.
- The carrier of the initial algebra for a functor  $F$  is its least fixpoint  $\mu F$ .
- If the ADT  $D$  is defined by  $D = F D$ , where  $F$  denotes the underlying functor  $F$  for  $D$ , then we interpret  $D$  as  $\mu F$ .

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# Transfinite Construction of Free Algebras (Kelly'80)

- If

$\mathcal{C}$  is a locally  $\lambda$ -presentable category interpreting types,

$0$  is the initial object of  $\mathcal{C}$ ,

and

$F : \mathcal{C} \rightarrow \mathcal{C}$  is a  $\lambda$ -cocontinuous functor

then  $F$  has an initial algebra, and its carrier is the least fixpoint  $\mu F$  of  $F$  computed by

$$0 \hookrightarrow F 0 \hookrightarrow F(F 0) \dots \hookrightarrow F^n 0 \dots \hookrightarrow \mu F$$

- I will be deliberately vague about the requirements needed on the category interpreting types and the functors underlying data types.
- For concreteness, take  $\mathcal{C}$  to be  $Set$  and  $F$  to be polynomial.

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# Semantics of ADTs



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data Bool : Set where
  false : Bool
  true  : Bool
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has  $F X = 1 + 1$ , so Bool is interpreted as  $\mu F$ , i.e., as  $\mu X. 1 + 1$



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has  $F X = 1 + A \times X$ , so List, A is interpreted as  $\mu F$ , i.e., as  $\mu X. 1 + A \times X$



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has  $F X = 1 + 1$ , so Bool is interpreted as  $\mu F$ , i.e., as  $\mu X. 1 + 1$



```
data Nat : Set where
  zero : Nat
  suc  : Nat → Nat
```

has  $F X = 1 + X$ , so Nat is interpreted as  $\mu F$ , i.e., as  $\mu X. 1 + X$



```
data List (A : Set) : Set where
  [] : List A
  _::_ : A → List A → List A
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data Tree (A : Set) (B : Set) : Set where
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# Semantics of ADTs



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data Bool : Set where
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  true  : Bool
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# Syntax of Nested Types (I)

Nested types encode stronger properties, leading to stronger correctness guarantees.

- Perfect trees

```
data PTree : Set → Set where
  pleaf  : ∀{A : Set} → A → PTree A
  pnode  : ∀{A : Set} → PTree(A × A) → PTree A
```

$\text{PTree } A$  encodes the constraint that a datum is a list of elements of type  $A$  whose length is a power of 2.

- Bushes

```
data Bush : Set → Set where
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- Constructors can have input types involving instances of the data type being defined other than the one being defined.
- For truly nested types like  $\text{Bush } A$  these instances can even involve themselves!
- The return type of every constructor is still the same (variable) instance of the data type as the one being defined.
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## Syntax of Nested Types (II)

- The general form of a nested type is

data  $D A_1 \dots A_n : B_1 \rightarrow \dots \rightarrow B_m \rightarrow \text{Set}$  where

$c_1 : \forall \{A_1 \dots A_n B_1 \dots B_m\} \rightarrow T_{11} \rightarrow \dots \rightarrow T_{1j_1} \rightarrow D A_1 \dots A_n B_1 \dots B_m$

...

$c_k : \forall \{A_1 \dots A_n B_1 \dots B_m\} \rightarrow T_{k1} \rightarrow \dots \rightarrow T_{kj_k} \rightarrow D A_1 \dots A_n B_1 \dots B_m$

where either

$T_{ij}$  is not inductive and does not mention  $D$

or

$T_{ij}$  is inductive and has the form

$$C_1 \rightarrow \dots \rightarrow C_p \rightarrow D A_1 \dots A_n V_1 \dots V_m$$

where  $D$  does not occur in any  $C_i$  or any  $V_i$ , and each  $V_i$  is functorial in  $B_1, \dots, B_m$

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  - $\implies$  no negative occurrences of  $D$  in argument types of constructors
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## Semantics of Nested Types (III)

- Like ADTs, nested types can be interpreted as fixpoints of functors...
  - ...but now the functors must be higher-order!
- If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then the collection of functors from  $\mathcal{C}$  to  $\mathcal{D}$  also form a category. Its objects are functors from  $\mathcal{C}$  to  $\mathcal{D}$  and its morphisms are natural transformations between such functors.
- A natural transformation  $\eta : F \rightarrow G$  is a collection  $\{\eta_X : F X \rightarrow G X\}_{X \in \mathcal{C}}$  such that if  $f : X \rightarrow Y$  in  $\mathcal{C}$  then  $\eta_Y \circ \text{map}_F f = \text{map}_G f \circ \eta_X$

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- A higher-order functor  $H$  is a functor (on a functor category) so it has an action on objects (functors) and on morphisms (natural transformations) of that category.
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- To give an initial algebra semantics for nested types we must compute fixpoints of higher-order functors.

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# Higher-Order Functorial Semantics of ADTs

- ADTs are uniform in their type parameters, so they also define inductive families.
- That is, we can interpret ADTs as fixpoints of higher-order functors too.

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data List (A : Set) : Set where
  []      : List A
  _ :: _ : A → List A → List A
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is also

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## maps for ADTs and Nested Types

- If the ADT or nested type  $D$  is defined by  $D = HD$ , where  $H$  denotes the higher-order functor  $H$  for  $D$ , then we interpret  $D$  as  $\mu H$ .
- Because  $\mu H$  is itself a functor, and thus has a corresponding  $map_{\mu H}$ ,  $D$  has a corresponding function  $map_D$  that is just the reflection back into syntax of  $map_{\mu H}$ .

- ```
mapList :: (A -> B) -> List A -> List B
mapList f []      = []
mapList f (x :: xs) = (f x) :: mapList f xs
```

- ```
mapPTree :: (A -> B) -> PTree A -> PTree B
mapPTree f (pleaf x)  = pleaf (f x)
mapPTree f (pnode ts) = pnode (mapPTree (f x f) ts)
```

- ```
mapBush :: (A -> B) -> Bush A -> Bush B
mapBush f bnil      = bnil
mapBush f (bnode a bb) = bnode (f a) (mapBush (mapBush f) bb)
```

- $map_D$  preserves the shape of a  $D$ -structure but (potentially) changes its contents.

## maps for ADTs and Nested Types

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- Because  $\mu H$  is itself a functor, and thus has a corresponding  $map_{\mu H}$ ,  $D$  has a corresponding function  $map_D$  that is just the reflection back into syntax of  $map_{\mu H}$ .

- - $map_{List} :: (A \rightarrow B) \rightarrow List\ A \rightarrow List\ B$   
 $map_{List}\ f\ [] = []$   
 $map_{List}\ f\ (x :: xs) = (f\ x) :: map_{List}\ f\ xs$

- - $map_{PTree} :: (A \rightarrow B) \rightarrow PTree\ A \rightarrow PTree\ B$   
 $map_{PTree}\ f\ (pleaf\ x) = pleaf\ (f\ x)$   
 $map_{PTree}\ f\ (pnode\ ts) = pnode\ (map_{PTree}\ (f \times f)\ ts)$

- - $map_{Bush} :: (A \rightarrow B) \rightarrow Bush\ A \rightarrow Bush\ B$   
 $map_{Bush}\ f\ bnil = bnil$   
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- $map_D$  preserves the shape of a  $D$ -structure but (potentially) changes its contents.

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$$\begin{aligned} \text{map}_{\text{Bush}} &:: (A \rightarrow B) \rightarrow \text{Bush } A \rightarrow \text{Bush } B \\ \text{map}_{\text{Bush}} f \text{bnil} &= \text{bnil} \\ \text{map}_{\text{Bush}} f (\text{bnode } a \text{ bb}) &= \text{bnode } (f a) (\text{map}_{\text{Bush}} (\text{map}_{\text{Bush}} f) \text{bb}) \end{aligned}$$

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# Naturality Results for ADTs and Nested Types (I)

- Just as their interpretations as fixpoints of higher-order functors give map functions for ADTs and nested types, these interpretations also give naturality results.
- A natural transformation  $\eta : \mu H \rightarrow \mu H'$  gives commuting squares: if  $f : X \rightarrow Y$ , then

$$\begin{array}{ccc} (\mu H) X & \xrightarrow{\eta X} & (\mu H') X \\ \text{map}_{\mu H} f \downarrow & & \downarrow \text{map}_{\mu H'} f \\ (\mu H) Y & \xrightarrow{\eta Y} & (\mu H') Y \end{array}$$

- Computationally (i.e., reflecting back into syntax), we can think of natural transformations as polymorphic functions between data types whose constructors are interpreted as  $\mu H$  and  $\mu H'$ .
- A polymorphic function (natural transformation) between (interpretations of) data types alters the shapes of data structures without changing their data elements.
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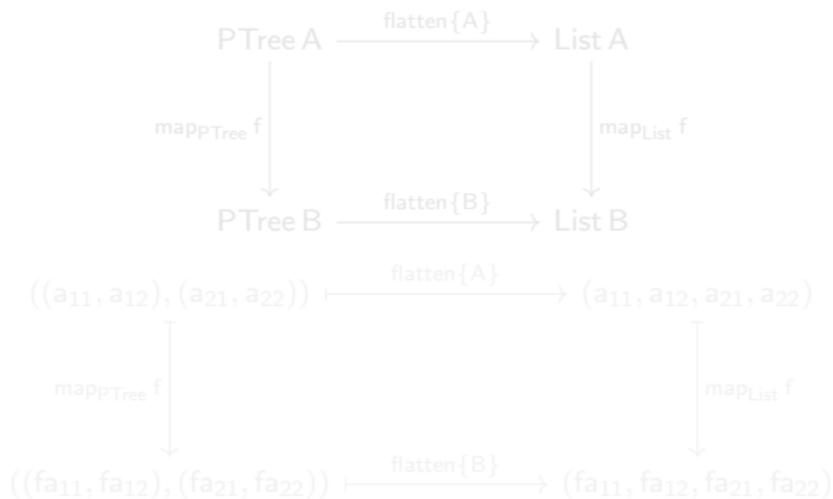
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## Naturality Results for ADTs and Nested Types (II)

- The naturality square for (the interpretation of) a polymorphic function says that it doesn't matter in which order we apply the function and the map operations.
- If the polymorphic function  $\text{flatten} : \forall\{A : \text{Set}\} \rightarrow \text{PTree } A \rightarrow \text{List } A$  acts like this

$$\text{flatten}(((a_{111}, a_{112}), (a_{121}, a_{122})), ((a_{211}, a_{212}), (a_{221}, a_{222}))) = (a_{111}, a_{112}, a_{121}, a_{122}, a_{211}, a_{212}, a_{221}, a_{222})$$

then



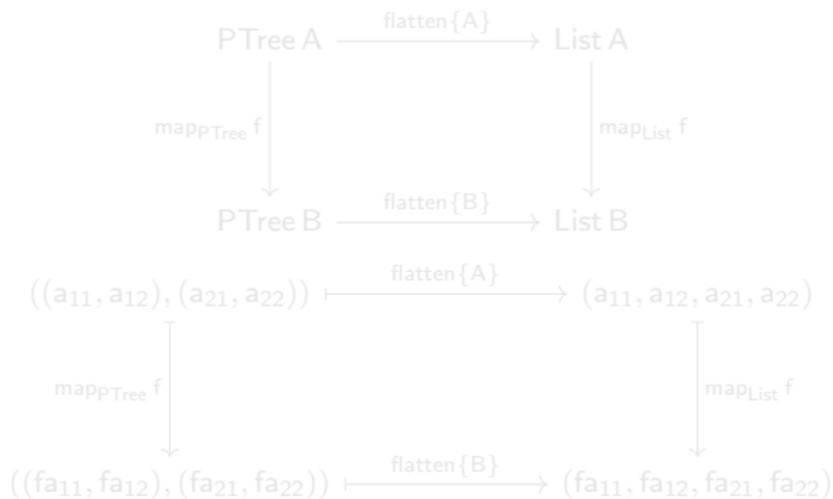
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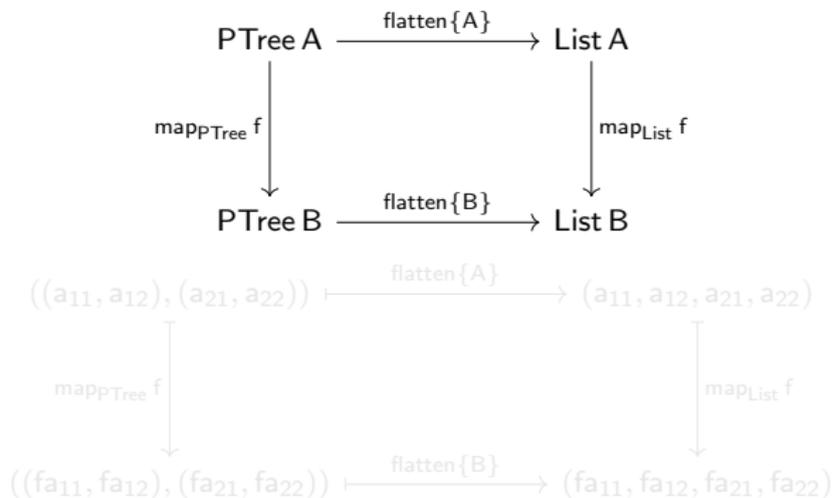
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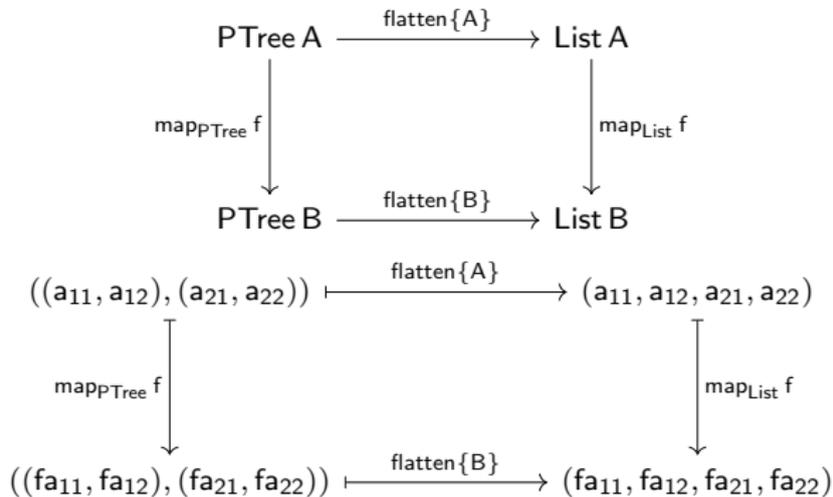
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# Summary

- Initial algebra semantics gives all of the above gives programming kit — maps, computation rules for polymorphic functions, folds (stylized recursion operators) — that we can use to program with, and reason about, ADTs and nested types.
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