Grundy Coloring
Chessboard Graphs

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Work sponsored by Appalachian State University and the Computing Research Association Committee on the Status of Women in Computing Research.
Grundy Coloring

• In a Grundy coloring, any vertex colored $k$ is adjacent to vertices colored $1 \ldots k-1$.

• The smallest number of colors possible in a Grundy coloring of graph $G$ is the same as the chromatic number, $\chi(G)$.

• The largest number of colors possible in a Grundy coloring of graph $G$ is called the Grundy number of $G$, $\Gamma(G)$. 
Grundy Coloring Example

• A bipartite graph on \( k \) vertices minus a matching gives us Grundy number of \( k \) and a chromatic number of 2!

• The Grundy number cannot be any larger than the maximum degree + 1, i.e., \( \Gamma(G) \leq \Delta(G) + 1 \).
Chessboard Graphs
Rooks – $n \times n$

- Every vertex on an $n \times n$ Rooks graph has the same number of neighbors ($2n - 2$).
- Thus, we would think that we could achieve a $2n - 1$ coloring. However, this is not the case.
Rooks – $n \times m$

- Every vertex on an $n \times m$ Rooks graph has the same number of neighbors ($n + m - 2$).
- This time we do achieve the maximum number of colors possible, $n + m - 1$.

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Knights Graph

• $\Delta(N_{n,m})$, the maximum degree of any vertex in an $n \times m$ Knights graph, is 8.

• What is the smallest board that can achieve a 9-coloring?
Knight’s Graph Grundy Results

- $\Gamma(N_6) = 7$
- $\Gamma(N_7) = 8$
- $\Gamma(N_8) = 8$?
- $\Gamma(N_{8,9}) = 9$

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Bishops – $n \times n$

- $\Delta(B_n)$, the maximum degree of any vertex in an $n \times n$ Bishops graph, is $2n - 2$.
- Conjecture: $\Gamma(B_n) = n + 2$ for $n > 5$. 

$$
\begin{array}{cccc}
1 & 4 & 4 & 4 \\
5 & 2 & 2 & 3 \\
3 & 6 & & 1 \\
1 & 7 & & 3 \\
3 & & 8 & 1 \\
1 & & 9 & 3 \\
3 & 2 & 2 & 10 \\
4 & 4 & 4 & 2 \\
\end{array}
$$
Bishops – $n \times m$, $m > n$

- $\Delta(B_{n,m}) = 2n - 2$.
- This maximum is only achievable when $m \geq 2n - 1$. 
Bishops – $n \times m$, $m > n$

$\Gamma(B_n) = 2n - 1$ when $m \geq 3n - 2$.

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Kings – $n \times m$

- The highest degree possible for any Kings graph with size at least $3 \times 3$ is 8.
Kings – $n \times m$, $m \geq n$

- Grundy coloring algorithm: In any $4 \times 6$ or larger graph, insert this block and fill in all empty squares with the smallest valid color for that square.

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 1 & 2 & 3 \\
3 & 4 & 9 & 7 & 5 & 1 \\
2 & 5 & 8 & 6 & 4 & 3 \\
3 & 1 & 3 & 2 & 1 & 2 \\
\end{array}
\]
Queens – $n \times n$

- $\Delta(Q_n) =$
  - $4n - 4$ if $n$ is odd.
  - $4n - 5$ if $n$ is even.

- Conjecture: $\Gamma(Q_n) \leq 2n + 3$. 
Queens – $n \times m$, $m \geq n$

- $\Delta(Q_{n,m}) = 3n + m - 4$
- It would seem possible to get $\Gamma(Q_{n,m}) \leq 3n + m - 3$; however, this is nowhere near achievable. . .
- Conjecture: $\Gamma(Q_{n,m}) \leq n + m + 3$
Queens Algorithms

• For the $2 \times m$ Queens board, 
  $\Gamma(Q_{2,m}) = 2 + m$, where $m \geq 6$. 

\[
\begin{array}{cccccccc}
5 & 9 & 8 & 1 & 2 & 3 & 4 \\
3 & 7 & 6 & 4 & 5 & 1 & 2 \\
\end{array}
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Queens Algorithms

- $Q_{3,m}$: $\Gamma(Q_{3,m}) = n + m + 1$, where $m \geq 8$. 

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Other colorings

• achromatic
• pseudo –achromatic
• partial Grundy