A Geometric Approach to Voting Theory

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Introduction

In this poster we highlight the connection between the way a voter's ranking of the candidates is scored, or weighted, and the outcome of an election, emphasizing that the selection of a weight system may strongly influence the outcome of an election. This observation is important from a practical standpoint in considering the objectivity of voting procedures. We then introduce and use geometric tools to analyze a new system that, while based on the idea of positional weighted voting, does not require the vote counter to choose particular values for the weights.

Counting the Votes

Positional weighted voting

In positional weighted voting each voter provides a full ranking of the candidates. The votes are counted by assigning a certain number of points–called a weight – to each first-place vote, a certain number of points to each second-place vote, etc.

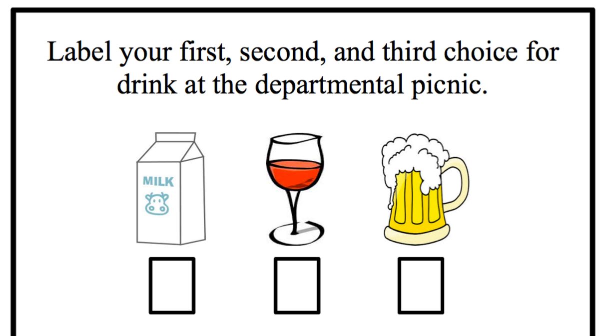


Figure 1: An example from Donald Saari [1]

We can organize the results of the vote of a 26-member department concisely as a linear system.

Points for Milk, Wine, and Beer

Weight's Vector

Milk's Row, Wine's Row, Beer's Row

$$\begin{bmatrix} 6 & 13 & 7 \\ 9 & 9 & 8 \\ 11 & 4 & 11 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} P_M \\ P_W \\ P_B \end{bmatrix}$$

1st Place Column

2nd Place Column

3rd Place Column

Determining a Winner

The choices of the w_i influence the election. Let $w_1 = 1$ and $w_3 = 0$. Popular choices for w_2 include:

- Plurality: $w_2 = 0$
- Borda count: $w_2 = 0.5$
- Anti-plurality: $w_2 = 1$

In our example if we let $w_1 = 1$ and $w_3 = 0$ we obtain the following linear system:

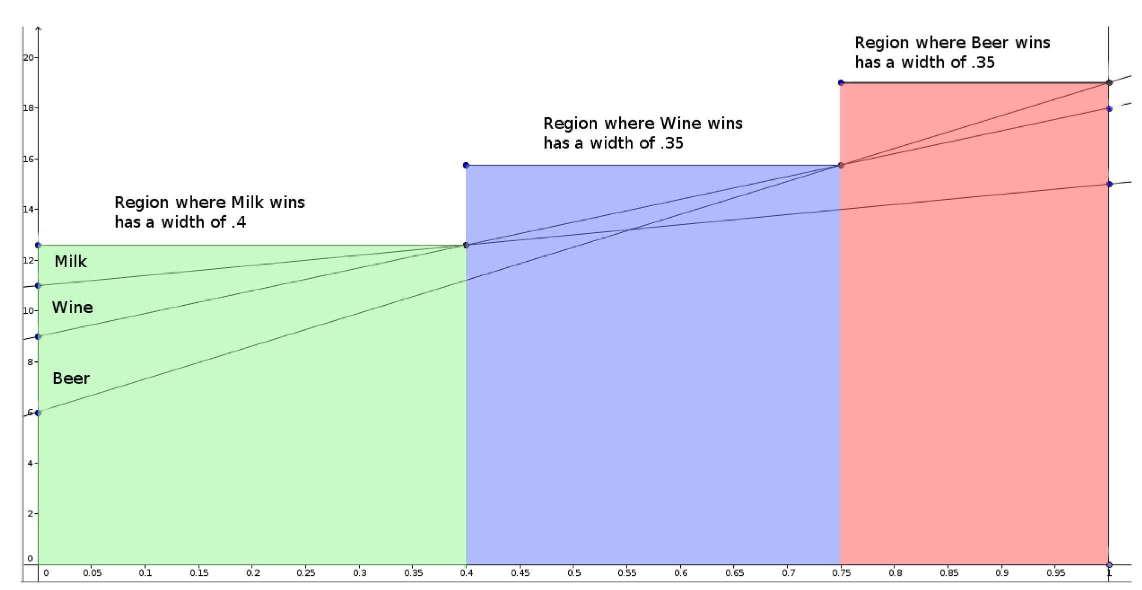


Figure 2: Total points earned vs. the weight w_2

- Plurality ($w_2 = 0$): milk wins
- Borda count ($w_2 = 0.5$): wine wins
- Anti-plurality ($w_2 = 1$): beer wins

Modified positional weighted voting with three candidates

In our new voting procedure, we let $w_1 = 1$, $w_3 = 0$, and allow w_2 to vary on the interval $[w_3, w_1]$. We call an interval $[a, b]$ over which a candidate receives more total points than any other candidate that candidate's *first place region* and the candidate whose first place region is the longest wins the election.

Figure 2 shows that milk would win our election under this new voting procedure.

Modified positional weighted voting with n candidates

In generalizing our three-candidate election to the n-candidate case we consider the linear system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}.$$

We fix $w_1 = 1$ and $w_n = 0$ and require $1 \geq w_2 \geq w_3 \cdots \geq w_{n-1} \geq 0$, creating an $(n-2)$-dimensional tetrahedron which we call *the reasonable weights space.* The first place regions of the candidates partition this space and the candidate with the first-place region of greatest volume wins the election.

A Four-Candidate Example

Consider this four-candidate election:

$$\begin{bmatrix} 11 & 7 & 3 & 9 \\ 10 & 8 & 4 & 8 \\ 9 & 10 & 2 & 9 \\ 0 & 5 & 21 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ w_2 \\ w_3 \\ 0 \end{bmatrix} = \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_D \end{bmatrix}$$

The system partitions the reasonable weight space as follows and elects candidate A as the winner.

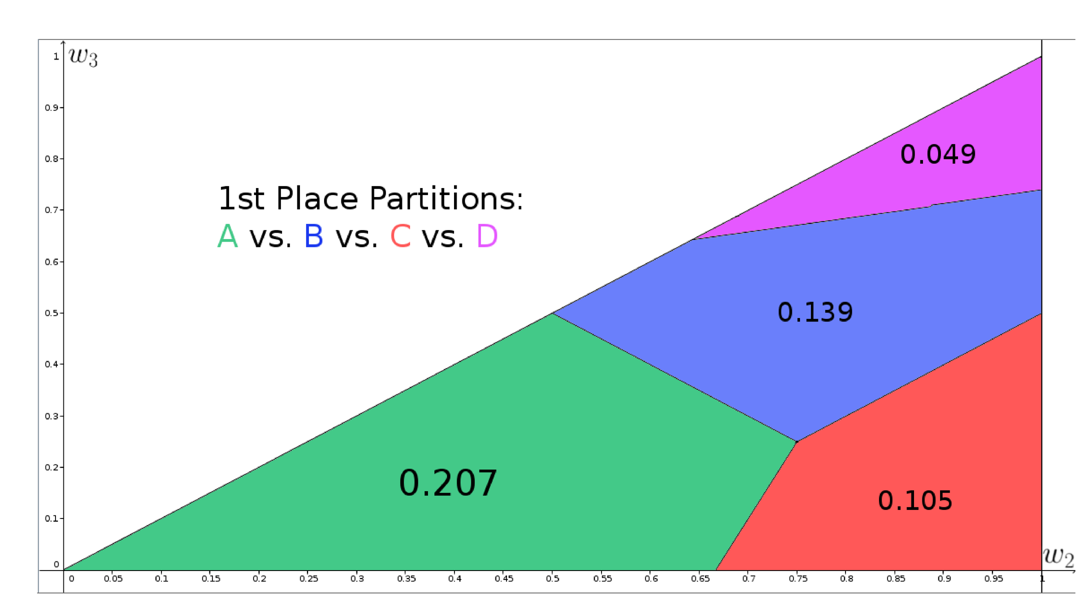


Figure 3: Winning regions graphed in the w_2 - w_3 plane.

Simplifying the Calculations

Consider this four-candidate election:

$$\begin{bmatrix} 1 & 11 & 1 & 0 \\ 8 & 1 & 1 & 3 \\ 3 & 0 & 5 & 5 \\ 1 & 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ w_2 \\ w_3 \\ 0 \end{bmatrix} = \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_D \end{bmatrix}$$

Looking at the corners of the reasonable weight space, we see B is preferred to C and D everywhere.

$$(w_2, w_3) = (0, 0) \to P_B > P_C > P_A = P_D$$
$$(w_2, w_3) = (1, 0) \to P_A > P_B > P_C > P_D$$
$$(w_2, w_3) = (1, 1) \to P_A > P_B > P_C > P_D$$

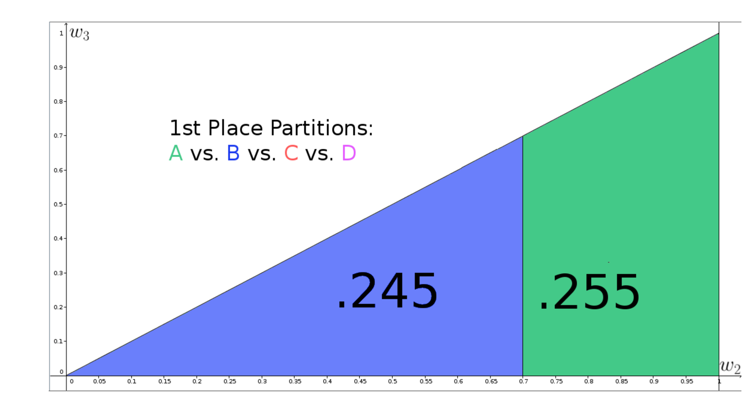
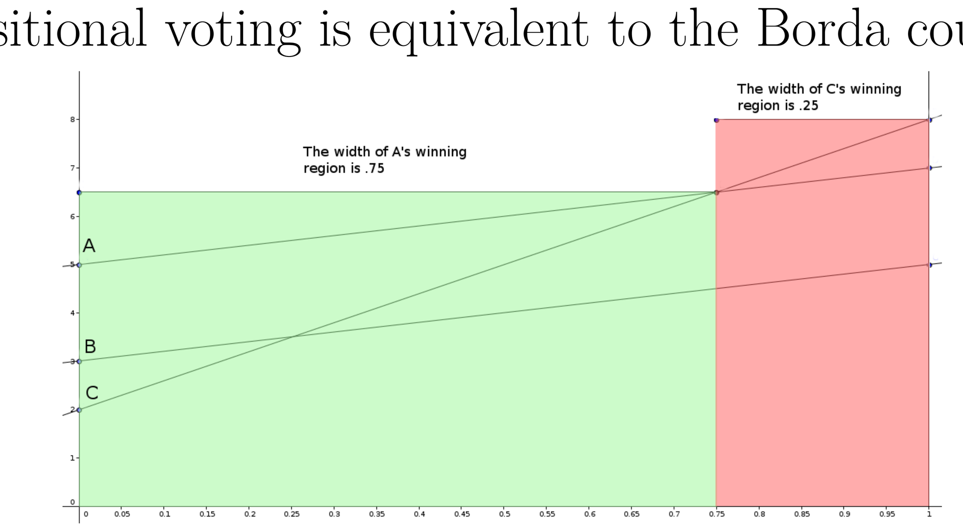


Figure 4: Now we find the winner by comparing A with B.

The corner theorem

If candidate X wins or ties against candidate Y at each corner of the reasonable weights space while strictly winning at at least one corner, X must rank higher than Y in the overall election and Y can be removed from consideration.

If $n = 3$ and the corner theorem applies, modified positional voting is equivalent to the Borda count.



References

[1] D. Saari & F. Valognes. Geometry, Voting, and Paradoxes. *Math. Mag.*, Vol. 71, No. 4 (Oct., 1998).

[2] Z. Daugherty, A. Eustis, G. Minton & M. Orrison. Voting, the Symmetric Group, and Representation Theory. *Amer. Math. Monthly*, Vol. 116, No. 8 (Oct., 2009).

*The author was partially supported by NSF grant DUE-1259496 as well as an Appalachian State University Undergraduate Research Assistantship